On spectrum of mesons in QCD2 t'Hooft model

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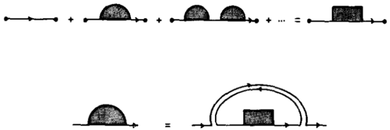
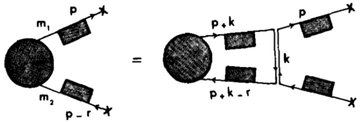
Introduction

In 4D QCD, color confinement is the phenomenon that color-charged particles - quarks and gluons cannot be directly observed. They must clump together to form hadrons: mesons and baryons.

Today we study toy models of the confinement, such as 2D QCD described by the following Lagrangian density

$$\mathcal{L}_{QCD_2} = -\frac{1}{2}\text{tr } F_{\mu\nu}F^{\mu\nu} + \sum_{k=1}^{N_f} \bar{\psi}_k(i\gamma^\mu D_\mu - m_k)\psi_k, \quad \mathcal{D}_\mu = \partial_\mu - ig_{YM}A_\mu.$$





In the planar limit $N_c \to 0$ with $g^2 = g_{YM}^2 N_c$ fixed, one finds that there are no free quarks in the theory, but an infinite tower of their bound states - mesons.
We will use the following notations

$$\alpha_i = \frac{\pi m_i^2}{g^2} - 1, \quad M_n^2 = 2\pi g^2 \lambda_n.$$

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'tHooft model (large N_c QCD)

$$2\pi^2\lambda_n\,\phi^{(n)}(x) = \mathcal{H}\phi^{(n)}(x) = \left[\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right]\phi^{(n)}(x) - \fint_0^1 dy\,\frac{\phi^{(n)}(y)}{(x-y)^2}.$$

Ising field theory

$$2\pi^2\lambda_n\phi^{(n)}(x) = \frac{\alpha}{x(1-x)}\phi^{(n)}(x) - \frac{1}{4}\int_0^1 \frac{dy(1-2x)(1-2y)}{2\sqrt{x(1-x)y(1-y)}}\phi^{(n)}(y) +$$
$$+\fint_0^1 \frac{dy}{2\sqrt{x(1-x)y(1-y)}}\,\frac{2xy-x-y}{(x-y)^2}\phi^{(n)}(y).$$

Large N_c scalar QCD

$$2\pi^2\lambda_n\,\phi^{(n)}(x) = \mathcal{H}\phi^{(n)}(x) = \left[\frac{\alpha_1}{x} + \frac{\alpha_2}{1-x}\right]\phi^{(n)}(x) - \fint_0^1 dy\,\frac{\phi^{(n)}(y)}{(x-y)^2}\,\frac{(x+y)(2-x-y)}{4x(1-x)}.$$

Introduction

In what follows, we introduce different parameters

$$\alpha = \frac{\alpha_1 + \alpha_2}{2}, \quad \beta = \frac{\alpha_2 - \alpha_1}{2},$$

so that the 't Hooft equation is rewritten as follows

$$2\pi^2 \lambda \, \phi(x|\beta) = \left[\frac{\alpha}{x(1-x)} + \frac{\beta \, (2x-1)}{x(1-x)} \right] \phi(x|\beta) - \fint_0^1 dy \, \frac{\phi(y|\beta)}{(x-y)^2}.$$

The replacement of $\beta \to -\beta$ leaves the spectrum λ_n invariant: $\lambda_n(\alpha, -\beta) = \lambda_n(\alpha, \beta)$

$$\phi^{(n)}(x|\beta) = (-1)^n \phi^{(n)}(1-x|-\beta).$$

The sign is conventional and is chosen to agree for $\beta = 0$ with another less trivial property of the eigenfunctions related to the parity symmetry. It reads

$$\sqrt{1+\alpha-\beta} \int_0^1 \frac{\phi^{(n)}(x|\beta)}{x} dx = (-1)^n \sqrt{1+\alpha+\beta} \int_0^1 \frac{\phi^{(n)}(x|\beta)}{1-x} dx.$$

Introduction

A convenient way to rewrite the 't Hooft equation further is in terms of the Fourier transform with respect to the new variable ν

$$\Psi(\nu|\beta) \overset{\text{def}}{=} \mathcal{F}[\phi(x|\beta)] = \int_0^1 \frac{dx}{2x(1-x)} \left(\frac{x}{1-x}\right)^{-\frac{i\nu}{2}} \phi(x|\beta)$$

In this variable 't Hooft equation has the form

$$\left(\frac{2\alpha}{\pi} + \nu \coth \frac{\pi\nu}{2}\right) \Psi(\nu|\beta) - \frac{2i\beta}{\pi} \fint_{-\infty}^{\infty} d\nu' \frac{1}{2\sinh \frac{\pi(\nu-\nu')}{2}} \Psi(\nu'|\beta) =$$

$$= \lambda \int_{-\infty}^{\infty} d\nu' \frac{\pi(\nu-\nu')}{2\sinh \frac{\pi(\nu-\nu')}{2}} \Psi(\nu'|\beta).$$

$$\Psi_n(\nu|\beta) = (-1)^n \Psi_n(-\nu|-\beta).$$

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It can be shown that $\Psi(\nu|\beta)$ is a meromorphic function of the complex variable ν with simple poles at

$$i\nu_k^*(\alpha+\beta)+2iN, \quad -i\nu_k^*(\alpha-\beta)-2iN, \quad N\geq 0,$$

where $i\nu_k^*(\alpha)$ are the roots of the transcendental equation

$$\frac{2\alpha}{\pi}+\nu\coth\left(\frac{\pi\nu}{2}\right)=0.$$

poles.eps

TQ equation and integrability

Define now the Q-function:

$$Q(\nu) \stackrel{\text{def}}{=} \left(\frac{2\alpha}{\pi}\sinh\left(\frac{\pi\nu}{2}\right) + \nu\cosh\left(\frac{\pi\nu}{2}\right)\right)\Psi(\nu) = \sinh\left(\frac{\pi\nu}{2}\right)\left(\frac{2\alpha}{\pi} + \nu\coth\frac{\pi\nu}{2}\right)\Psi(\nu)$$

and consider its meromorphic continuation to the maximal domain of analyticity.

The integral equation for $\Psi(\nu)$ and the finiteness of the norm imply the following analytic properties of the function $Q(\nu)$ valid in the strip $\mathrm{Im}\,\nu \in [-2, 2]$:

1. it grows slower than any exponential as $|\mathrm{Re}\,\nu| \to \infty$, meaning that it remains bounded as
$$\forall\epsilon > 0 \quad Q(\nu) = \mathcal{O}(e^{\epsilon|\nu|}), \quad |\mathrm{Re}\,\nu| \to \infty,$$
2. it necessarily has
$$\text{zeroes}: \quad 0, \quad \pm 2i, \quad \pm i\nu_1^*(\alpha)$$
3. and might have
$$\text{poles:} \qquad i\nu_1^*(\alpha+\beta), \quad -i\nu_1^*(\alpha-\beta).$$

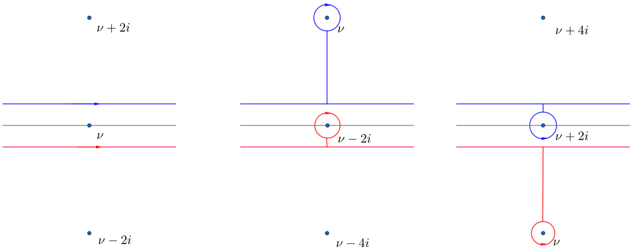
TQ equation and integrability

It can be shown that $Q(\nu)$ satisfies Baxter's TQ equation

$$\left(1+\frac{\beta x}{\nu+2i+\alpha x}\right)Q(\nu+2i)+\left(1-\frac{\beta x}{\nu-2i+\alpha x}\right)Q(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q(\nu),$$

where

$$x=\frac{2}{\pi}\tanh\left(\frac{\pi\nu}{2}\right), \quad z=2\pi\lambda\tanh\left(\frac{\pi\nu}{2}\right).$$



TQ equation and integrability

On the other hand, assume that we have a solution of the TQ equation which admits the required analyticity properties, but the conditions $Q(0) = Q(\pm 2i) = 0$ are relaxed. Then the corresponding function $Q(\nu) = \sinh\left(\frac{\pi\nu}{2}\right)\left(\frac{2\alpha}{\pi} + \nu \coth\frac{\pi\nu}{2}\right)\Psi(\nu)$ satisfies an inhomogeneous equation

$$\left(\frac{2\alpha}{\pi} + \nu \coth\frac{\pi\nu}{2}\right)\Psi(\nu|\beta) - \frac{2i\beta}{\pi} \fint_{-\infty}^{\infty} d\nu' \frac{1}{2\sinh\frac{\pi(\nu-\nu')}{2}} \Psi(\nu'|\beta) =$$

$$= \lambda \fint_{-\infty}^{\infty} d\nu' \frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}} \Psi(\nu'|\beta) + c_+ e_+(\nu) + c_- e_-(\nu),$$

where

$$e_+(\nu) \stackrel{\text{def}}{=} \frac{\nu}{\sinh\frac{\pi\nu}{2}}, \quad e_-(\nu) \stackrel{\text{def}}{=} \frac{1}{\sinh\frac{\pi\nu}{2}}$$

and

$$c_+ = \frac{Q(2i) - Q(-2i) + \frac{2\beta}{1+\alpha}Q(0)}{4i}, \quad c_- = Q(0).$$

TQ equation and integrability

Our strategy will now be as follows. For the TQ equation

$$\left(1+\frac{\beta x}{\nu+2i+\alpha x}\right)Q_\pm(\nu+2i)+\left(1-\frac{\beta x}{\nu-2i+\alpha x}\right)Q_\pm(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q_\pm(\nu),$$

and, consequently, for the inhomogeneous equation

$$\left(\frac{2\alpha}{\pi}+\nu\coth\frac{\pi\nu}{2}\right)\Psi_\pm(\nu|\beta)-\frac{2i\beta}{\pi}\int_{-\infty}^{\infty}d\nu'\,\frac{1}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi_\pm(\nu'|\beta)=$$

$$=\lambda\int_{-\infty}^{\infty}d\nu'\,\frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi_\pm(\nu'|\beta)+e_\pm(\nu)$$

the solution exists and is unique. The $Q-$functions have the following symmetry properties

$$Q_\pm(\nu|\alpha,\beta)=\mp Q_\pm(-\nu|\alpha,-\beta)$$

and satisfy the normalization conditions

$$Q_+(0|\alpha,\beta)=0,\quad Q_+(2i|\alpha,\beta)-Q_+(-2i|\alpha,\beta)=4i,$$

$$Q_-(0|\alpha,\beta)=1,\quad Q_-(2i|\alpha,\beta)-Q_-(-2i|\alpha,\beta)=-\frac{2\beta}{1+\alpha}.$$

TQ equation and integrability

There are two basic questions:

- How to solve inhomogeneous integral equation?
- How to extract the spectral data?

In fact 't Hooft integral equation for $\beta = 0$ belongs to the class of Frehholm integral equations with integrable kernel (Its-Izergin-Korepin-Slavnov 1990).

But it is even better since it can be rewritten in terms of TQ-equation which admits in a sense an exact solution. It was noticed for $\alpha = \beta = 0$ by Fateev, Lukyanov and Zamolodchikov in 2009.

As for the second question, it is clear that at $\lambda \to \lambda_n$ one should recover the homogeneous equation. Thus qualitatively

$$Q_{\pm}(\nu, \lambda) = \sum_{n=0}^{\infty} \frac{c_n^{\pm} Q_n(\nu)}{\lambda - \lambda_n},$$

Solution of TQ equation in the small λ regime

We look for a solution of the TQ equation

$$\left(1+\frac{\beta x}{\nu+2i+\alpha x}\right)Q(\nu+2i)+\left(1-\frac{\beta x}{\nu-2i+\alpha x}\right)Q(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q(\nu),$$

$$x=\frac{2}{\pi}\tanh\left(\frac{\pi\nu}{2}\right),\quad z=2\pi\lambda\tanh\left(\frac{\pi\nu}{2}\right).$$

as a formal power series

$$Q_\pm(\nu,\lambda)=\sum_{k=0}^{\infty}Q_\pm^{(k)}(\nu)\lambda^k,$$

Remarkably, there are two exact solutions

$$\Xi(\nu|\alpha,\beta)=(\nu+\alpha x)\,F\left(\begin{matrix}1+\frac{i(\nu+(\alpha-\beta)x)}{2}\\ 2-i\beta x\end{matrix}\middle|-iz\right),$$

$$\Sigma(\nu|\alpha,\beta)=\frac{\Gamma\left(\frac{i(\nu+(\alpha+\beta)x)}{2}\right)}{\Gamma\left(1+\frac{i(\nu+(\alpha-\beta)x)}{2}\right)}(\nu+\alpha x)\,F\left(\begin{matrix}\frac{i(\nu+(\alpha+\beta)x)}{2}\\ i\beta x\end{matrix}\middle|-iz\right),$$

Solution of TQ equation in the small λ regime

First, we note that the Γ factor in the function $\Sigma(\nu|\alpha,\beta)$ exhibits an unpleasant behavior at the points $\nu = \pm i$ where $x(\nu)$ has poles. In order to restore the analyticity we redefine $\Sigma(\nu|\alpha,\beta)$ by replacing

$$\frac{\Gamma\left(\frac{i(\nu+(\alpha+\beta)x)}{2}\right)}{\Gamma\left(1+\frac{i(\nu+(\alpha-\beta)x)}{2}\right)} \quad \longrightarrow \quad 2\pi^2 x \frac{S_0(\nu|\alpha+\beta)}{\nu+(\alpha+\beta)x}\frac{S_0(-\nu|\alpha-\beta)}{\nu+(\alpha-\beta)x},$$

where the function $S_0(\nu|\alpha)$ is defined by the integral

$$S_0(\nu+i|\alpha) = \exp\left[\frac{i}{4}\int_{-\infty}^{\infty} \log\left(\frac{4\pi\tanh\left(\frac{\pi t}{2}\right)}{t+\frac{2\alpha}{\pi}\tanh\left(\frac{\pi t}{2}\right)}\right)\left(\tanh\frac{\pi(t-\nu)}{2} - \tanh\frac{\pi t}{2}\right) dt\right]$$

in the strip $\mathrm{Im}\ \nu \in [0,2]$ and by analytic continuation elsewhere.
This function satisfies the same shift relation as the Γ function

$$S_0(\nu+2i|\alpha) = \frac{4\pi\tanh\left(\frac{\pi\nu}{2}\right)}{\nu+\frac{2\alpha}{\pi}\tanh\left(\frac{\pi\nu}{2}\right)} S_0(\nu|\alpha),$$

It can be shown that $S_0(\nu)$ is a unique solution analytic in the strip $\mathrm{Im}\ \nu \in [0,2]$ (in fact, it is analytic in the larger domain $\mathrm{Im}\ \nu \in [-2,2]$ except for one point $\nu = -2i$).

Solution of TQ equation in the small λ regime

Given a solution of TQ equation, even one lacking required analyticity or symmetry properties, one can multiply it by an arbitrary $2i$ periodic function of ν (quasiconstant) and obtain a new solution. For example mutliplying by $e^{\frac{iz}{2}}$, one obtains the solutions

$$M_+(\nu|\alpha,\beta) = e^{\frac{iz}{2}}\Xi(\nu|\alpha,\beta), \quad M_-(\nu|\alpha,\beta) = e^{\frac{iz}{2}}\Sigma(\nu|\alpha,\beta),$$

which can be shown to satisfy the symmetry properties

$$M_\pm(-\nu|\alpha,-\beta) = \mp M_\pm(\nu|\alpha,\beta).$$

But they still lack the required analiticity properties: in both hypergeometric function

$$F\left(\begin{matrix} 1+\frac{i(\nu+(\alpha-\beta)x)}{2} \\ 2-i\beta x \end{matrix}\middle| -iz\right) \quad \text{and} \quad F\left(\begin{matrix} \frac{i(\nu+(\alpha+\beta)x)}{2} \\ i\beta x \end{matrix}\middle| -iz\right),$$
$$x = \frac{2}{\pi}\tanh\left(\frac{\pi\nu}{2}\right), \quad z = 2\pi\lambda\tanh\left(\frac{\pi\nu}{2}\right).$$

there are poles of growing order at $\nu = \pm i$ that appear in the λ expansion and the singularities coming from the zeros of the Pochhammer symbols.

Solution of TQ equation in the small λ regime

We look for solutions of TQ-equation in a more general form

$$Q_{\pm}(\nu|\alpha,\beta) = A_{\pm}(x|\lambda)M_{\pm}(\nu|\alpha,\beta) + B_{\pm}(x|\lambda)M_{\mp}(\nu|\alpha,\beta),$$

where $A_{\pm}(x|\lambda)$ and $B_{\pm}(x|\lambda)$ admit the expansion in the parameter λ. These solutions have to satisfy the generalized symmetry relations

$$Q_{\pm}(-\nu|\alpha,-\beta) = \mp Q_{\pm}(\nu|\alpha,\beta),$$

as well as the normalization conditions

$$Q_{+}(2i|\alpha,\beta) = 2i, \qquad Q_{+}(0|\alpha,\beta) = 0,$$
$$Q_{-}(0|\alpha,\beta) = 1, \qquad Q_{-}(2i|\alpha,\beta) - Q_{-}(-2i|\alpha,\beta) = -\frac{2\beta}{1+\alpha}.$$

The role of the functions $A_{\pm}(x|\lambda)$ and $B_{\pm}(x|\lambda)$ is to cancel poles at $\nu = \pm i$ and at

$$i\beta x_k = \frac{2i\beta}{\pi}\tanh\left(\frac{\pi\nu_k}{2}\right) = k, \quad k \in \mathbb{Z}, \quad k \neq 1.$$

We note that in the strip $\mathrm{Im}\,\nu \in [-2,2]$ for $\beta > 0$ there are two solutions

$$\nu_k = -\frac{2i}{\pi}\arctan\frac{\pi k}{2\beta} \quad \text{and} \quad \nu_k = -\frac{2i}{\pi}\arctan\frac{\pi k}{2\beta} + 2i \quad \text{for} \quad k > 0,$$
$$\nu_k = -\frac{2i}{\pi}\arctan\frac{\pi k}{2\beta} \quad \text{and} \quad \nu_k = -\frac{2i}{\pi}\arctan\frac{\pi k}{2\beta} - 2i \quad \text{for} \quad k < 0.$$

Solution of TQ equation in the small λ regime

In the following, we will use the notations

$$\arctan \frac{\pi k}{2\beta} = \xi_k \quad k > 0,$$

and

$$g_k \stackrel{\text{def}}{=} G(\nu_k), \quad G(\nu|\alpha,\beta) \stackrel{\text{def}}{=} \frac{S_0(\nu|\alpha+\beta)}{S_0(\nu|\alpha-\beta)},$$

$$S_0(\nu+i|\alpha) = \exp\left[\frac{i}{4}\int_{-\infty}^{\infty} \log\left(\frac{4\pi\tanh\left(\frac{\pi t}{2}\right)}{t+\frac{2\alpha}{\pi}\tanh\left(\frac{\pi t}{2}\right)}\right)\left(\tanh\frac{\pi(t-\nu)}{2} - \tanh\frac{\pi t}{2}\right)dt\right].$$

We note that g_k and g_{-k} are not independent, but obey the following relation

$$g_k g_{-k} = \frac{2\beta\xi_k + \pi k(\alpha+\beta)}{2\beta\xi_k + \pi k(\alpha-\beta)} \quad \text{for} \quad k > 0.$$

Solution of TQ equation in the small λ regime

For $A_\pm(x|\lambda)$ and $B_\pm(x|\lambda)$ in

$$Q_\pm(\nu|\alpha,\beta) = A_\pm(x|\lambda)M_\pm(\nu|\alpha,\beta) + B_\pm(x|\lambda)M_\mp(\nu|\alpha,\beta),$$

we take the following ansatz

$$A_+(x|\lambda) = \frac{1}{1-i\beta x} + \frac{a_+^{(1)}x}{1-i\beta x}\lambda + \frac{a_+^{(2)}x + a_+^{(3)}x^2 + a_+^{(4)}x^3}{(1-i\beta x)(1+i\beta x)}\lambda^2 + \dots$$

$$B_+(x|\lambda) = \frac{b_+^{(1)}x}{1-i\beta x} + \frac{b_+^{(2)}x + b_+^{(3)}x^2 + b_+^{(4)}x^3}{(1-i\beta x)(2-i\beta x)}\lambda + \frac{b_+^{(5)}x + b_+^{(6)}x^2 + b_+^{(7)}x^3 + b_+^{(8)}x^4 + b_+^{(9)}x^5}{(1-i\beta x)(2-i\beta x)(3-i\beta x)}\lambda^2$$

and

$$A_-(x|\lambda) = a_-^{(0)} + \frac{a_-^{(1)}x + a_-^{(2)}x^2}{1-i\beta x}\lambda + \frac{a_-^{(3)}x + a_-^{(4)}x^2 + a_-^{(5)}x^3 + a_-^{(6)}x^4}{(1-i\beta x)(2-i\beta x)}\lambda^2 + \dots$$

$$B_-(x|\lambda) = \frac{b_-^{(1)}}{1-i\beta x}\lambda + \frac{b_-^{(2)}x + b_-^{(3)}x^2}{(1-i\beta x)(1+i\beta x)}\lambda^2 + \frac{b_-^{(4)}x + b_-^{(5)}x^2 + b_-^{(6)}x^3 + b_-^{(7)}x^4}{(1-i\beta x)(1+i\beta x)(2+i\beta x)}\lambda^2 + \dots$$

where the coefficients $a_\pm^{(s)}$ and $b_\pm^{(s)}$ are to be adjusted in a way to cancel all the unwanted poles ($\nu = \pm i$ and Pochhammer poles).

Solution of TQ equation in the small λ regime

Explicit expressions for these coefficients are rather cumbersome. For example

$$a_+^{(1)} = -\frac{i\pi^2\alpha}{2\beta}, \quad b_+^{(1)} = -\frac{\pi(\alpha+\beta)+2\beta\xi_1}{\pi g_1},$$
$$a_+^{(2)} = \frac{i\pi^2\left(2\beta\xi_1+\pi(\alpha+\beta)\right)^2}{8\beta^3 g_1^2} - \frac{i\pi^4\left(\alpha^2-\beta^2\right)}{8\beta^3},$$
$$b_+^{(2)} = \frac{2\pi\alpha\left(2\beta\xi_1+\pi(\alpha+\beta)\right)}{\beta^2 g_1} - \frac{2\left(\beta\xi_2+\pi\alpha\right)\left(\beta\xi_2+\pi(\alpha+\beta)\right)}{\beta^2 g_2}$$

and

$$a_-^{(0)} = \frac{\gamma}{1+\alpha}, \quad a_-^{(1)} = \frac{i\pi^2\alpha\gamma}{2\beta(1+\alpha)} - \frac{i\pi\left(2\beta\xi_1+\pi(\alpha+\beta)\right)}{2\beta g_1}, \quad b_-^{(1)} = \frac{i\pi^2}{2\beta},$$
$$a_-^{(2)} = \frac{\pi^2\alpha\gamma}{2(1+\alpha)}, \quad b_-^{(2)} = \frac{\pi^4\alpha}{4\beta^2} - \frac{\pi^3\gamma\left(2\beta\xi_1+\pi(\alpha+\beta)\right)}{4\beta^2 g_1(1+\alpha)},$$

where we have introduced

$$\gamma \overset{\text{def}}{=} \sqrt{(1+\alpha+\beta)(1+\alpha-\beta)}.$$

Solution of TQ equation in the small λ regime

Summarizing, we have constructed solutions of the TQ equation

$$\left(1+\frac{\beta x}{\nu+2i+\alpha x}\right)Q(\nu+2i)+\left(1-\frac{\beta x}{\nu-2i+\alpha x}\right)Q(\nu-2i)-2Q(\nu)=-\frac{2z}{\nu+\alpha x}Q(\nu),$$

or of the inhomogeneous integral equation ($Q(\nu)=\sinh\left(\frac{\pi\nu}{2}\right)\left(\frac{2\alpha}{\pi}+\nu\coth\frac{\pi\nu}{2}\right)\Psi(\nu)$)

$$\left(\frac{2\alpha}{\pi}+\nu\coth\frac{\pi\nu}{2}\right)\Psi_\pm(\nu)-\frac{2i\beta}{\pi}\fint_{-\infty}^{\infty}d\nu'\,\frac{1}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi_\pm(\nu')=$$

$$=\lambda\fint_{-\infty}^{\infty}d\nu'\,\frac{\pi(\nu-\nu')}{2\sinh\frac{\pi(\nu-\nu')}{2}}\Psi_\pm(\nu')+e_\pm(\nu)$$

as small λ series

$$Q_\pm(\nu,\lambda)=\sum_{k=0}^{\infty}Q_\pm^{(k)}(\nu)\lambda^k,$$

Each coefficient $Q_\pm^{(k)}(\nu)$ contains the transcendental part $G(\nu)$ and then polynomially depends on ν and rationally on $x=\frac{2}{\pi}\tanh\frac{\pi\nu}{2}$ with coefficients being rational functions of transcendents g_k.

Extracting the spectral data

Having found the solutions $Q_{\pm}(\nu, \lambda)$ to TQ-equation, one might try to find the spectrum from

$$Q_{\pm}(\nu, \lambda) = \sum_{n=0}^{\infty} \frac{c_n^{\pm} Q_{(\pm)}^{(n)}(\nu)}{\lambda - \lambda_n},$$

where $Q_{(\pm)}^{(n)}(\nu)$ are normalized eigenfunctions. The problem is that we don't know neither $Q_{(\pm)}^{(n)}(\nu)$, nor $c_n^{\pm}$.

Some new information can be gained by considering the "quantum Wronskian"

$$W(\nu) \stackrel{\text{def}}{=} Q_+(\nu + i) Q_-(\nu - i) - Q_-(\nu + i) Q_+(\nu - i).$$

It is a direct consequence of the TQ equation that $W(\nu)$ satisfies

$$W(\nu + i) = \frac{1 - \frac{\beta x}{\nu - 2i + \alpha x}}{1 + \frac{\beta x}{\nu + 2i + \alpha x}} W(\nu - i) = \frac{\nu - 2i + (\alpha - \beta)x}{\nu - 2i + \alpha x} \frac{\nu + 2i + \alpha x}{\nu + 2i + (\alpha + \beta)x} W(\nu - i).$$

which has a unique solution

$$W(\nu + i) = 2i \frac{\gamma}{1 + \alpha} \frac{\nu + \alpha x}{\nu + (\alpha + \beta)x} \frac{\nu + 2i + \alpha x}{\nu + 2i + (\alpha + \beta)x} G(\nu).$$

Extracting the spectral data

In particular we see that the quantum Wronskian does not depend on λ, although $Q_\pm(\nu)$ are expected to be singular when $\lambda \to \lambda_n$. Consider its value at $\nu = 0$

$$W(0) = Q_+(i)Q_-(-i) - Q_+(-i)Q_-(i) = 2i\frac{\gamma}{1+\alpha}\frac{\alpha^2}{\alpha^2-\beta^2},$$
$$\gamma = \sqrt{(1+\alpha+\beta)(1+\alpha-\beta)}$$

Since

$$Q_\pm(\nu|\alpha,\beta) = \mp Q_\pm(-\nu|\alpha,-\beta)$$

we have for $\beta = 0$

$$Q_+(i,\beta=0)Q_-(i,\beta=0) = i.$$

On the other hand, for $\beta = 0$

$$Q_+(\nu,\beta=0) = \sum_{n=0}^{\infty}\frac{c_n^+ Q_+^{(2n)}(\nu)}{\lambda-\lambda_{2n}}, \quad Q_-(\nu,\beta=0) = \sum_{n=0}^{\infty}\frac{c_n^- Q_+^{(2n-1)}(\nu)}{\lambda-\lambda_{2n-1}}.$$

Combining both, we expect that

$$Q_+(i,\beta=0) \sim \prod_{n=1}^{\infty}\frac{\lambda-\lambda_{2n-1}}{\lambda-\lambda_{2n}}, \quad Q_-(i,\beta=0) \sim \prod_{n=1}^{\infty}\frac{\lambda-\lambda_{2n}}{\lambda-\lambda_{2n-1}}$$

Extracting the spectral data

It is convenient to define spectral determinants (Fredholm determinants). They are the generating functions of the even/odd spectral sums, defined respectively as the traces over even/odd states:

$$\mathcal{G}_+^{(s)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{2n}^s} - \frac{\delta_{s,1}}{n+1} \right), \quad \mathcal{G}_-^{(s)} \stackrel{\text{def}}{=} \sum_{n=0}^{\infty} \left(\frac{1}{\lambda_{2n+1}^s} - \frac{\delta_{s,1}}{n+1} \right).$$

so that

$$\mathcal{D}_\pm(\lambda) = \left(\frac{2\pi}{e}\right)^\lambda \exp\left[-\sum_{s=1}^{\infty} s^{-1} \mathcal{G}_\pm^{(s)} \lambda^s \right].$$

are given by the convergent products

$$\mathcal{D}_+(\lambda) \stackrel{\text{def}}{=} \left(\frac{2\pi}{e}\right)^\lambda \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n}}\right) e^{\frac{\lambda}{n+1}}, \quad \mathcal{D}_-(\lambda) \stackrel{\text{def}}{=} \left(\frac{2\pi}{e}\right)^\lambda \prod_{n=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_{2n+1}}\right) e^{\frac{\lambda}{n+1}}.$$

Extracting the spectral data

For $\beta = 0$ one has

$$\underbrace{\overbrace{\left(\frac{2\alpha}{\pi} + \nu \coth \frac{\pi\nu}{2}\right)}^{f(\nu)} \Psi(\nu)}_{\psi(\nu)} = \lambda \int_{-\infty}^{\infty} \frac{\pi(\nu - \nu')}{2\sinh \frac{\pi(\nu-\nu')}{2}} \Psi(\nu') d\nu'.$$

takes the form

$$\psi(u) = \lambda \int_{\mathbb{R}} K(u, v)\psi(v) dv,$$

where (here $M(u) = -\coth \frac{\pi u}{2}$)

$$K(u, v) = \frac{\pi(u - v)}{2\sinh \frac{\pi(u-v)}{2}} \frac{1}{f(v)} = \frac{e_+(u)e_-(v) - e_-(v)e_+(u)}{M(u) - M(v)} \frac{1}{f(v)}.$$

Then the kernel of the resolvent $R = \frac{K}{1-\lambda K}$ can be expressed in a similar way (Its-Isergin-Korepin-Slavnov)

$$R(u, v) = \frac{\psi_+(u)\psi_-(v) - \psi_-(v)\psi_+(u)}{M(u) - M(v)} \frac{1}{f(v)},$$

where

$$\psi_\pm(u) = \lambda \int_{\mathbb{R}} K(u, v)\psi_\pm(v) + e_\pm(v).$$

Extracting the spectral data

The spectral determinants can be expressed in terms of the resolvent as follows

$$\partial_\lambda \log(\mathcal{D}_+(\lambda)\mathcal{D}_-(\lambda)) = \int_{\mathbb{R}}' R(v,v)dv, \quad \partial_\lambda \log\left(\frac{\mathcal{D}_+(\lambda)}{\mathcal{D}_-(\lambda)}\right) = \int_{\mathbb{R}} R(v,-v)dv$$

Then one can prove that

$$\frac{Q_+(i,\beta=0)}{Q_+(2i,\beta=0)} = \frac{1}{2}\frac{\mathcal{D}_-(\lambda)}{\mathcal{D}_+(\lambda)} \quad \text{and} \quad \frac{Q_-(i,\beta=0)}{Q_-(0,\beta=0)} = \frac{\mathcal{D}_+(\lambda)}{\mathcal{D}_-(\lambda)}.$$

Indeed (remember that $Q_\pm(u) = \sinh \frac{\pi u}{2}\psi_\pm(u)$)

$$\psi_\pm = \frac{1}{1-\lambda K}e_\pm \implies \partial\psi_\pm = \frac{K}{(1-\lambda K)^2}e_\pm = \underbrace{\frac{K}{1-\lambda K}}_{R}\underbrace{\frac{1}{1-\lambda K}e_\pm}_{\psi_\pm}$$

$$\partial\psi_\pm(u) = \int_{\mathbb{R}} R(u,v)\psi_\pm(v)dv = \int_{\mathbb{R}} \frac{\psi_+(u)\psi_-(v)-\psi_-(v)\psi_+(u)}{M(u)-M(v)}\frac{1}{f(v)}\psi_\pm(v)dv$$

Since $\psi_\pm(-v) = \pm\psi_\pm(v)$ and $M(i)=0$, one has

$$\partial\psi_\pm(i) = \pm\psi_\pm(i)\int_{\mathbb{R}} \frac{\psi_+(v)\psi_-(v)}{M(v)f(v)}dv = \pm\psi_\pm(i)\int_{\mathbb{R}} R(v,-v)dv = \pm\psi_\pm(i)\partial_\lambda \log\left(\frac{\mathcal{D}_+(\lambda)}{\mathcal{D}_-(\lambda)}\right)$$

Extracting the spectral data

The relations

$$\frac{Q_+(i,\beta=0)}{Q_+(2i,\beta=0)} = \frac{1}{2}\frac{\mathcal{D}_-(\lambda)}{\mathcal{D}_+(\lambda)} \quad \text{and} \quad \frac{Q_-(i,\beta=0)}{Q_-(0,\beta=0)} = \frac{\mathcal{D}_+(\lambda)}{\mathcal{D}_-(\lambda)}.$$

are not enough to study $D_+(\lambda)$ and $D_-(\lambda)$ individually. In fact, there are identities of differential type

$$\begin{aligned} \partial_\lambda \log D_-(\lambda) - \frac{\alpha}{4}\mathtt{i}_2(\alpha) &= 2i\,\partial_\nu \log Q_+(\nu)\Big|_{\nu=i}, \\ \partial_\lambda \log D_+(\lambda) - \frac{\alpha}{4}\mathtt{i}_2(\alpha) &= 2i\left(1-\frac{2\alpha}{\pi^2}\lambda^{-1}\right)\partial_\nu \log Q_-(\nu)\Big|_{\nu=i}, \end{aligned} \qquad (*)$$

where the constant term is

$$\mathtt{i}_{2k}(\alpha) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\sinh t(\sinh 2t - 2t)}{t\cosh^{2k} t\big(\alpha\sinh t + t\cosh t\big)}dt.$$

The relations (*) is a conjecture!

Extracting the spectral data

Using (*), one can find analytic expressions for the spectral sums. For example

$$G_+^{(1)} = \log(8\pi) - 1 - \frac{7\alpha\zeta(3)}{\pi^2} - \frac{\alpha}{2}\left(\mathrm{u}_1(\alpha) - \alpha \mathrm{u}_3(\alpha)\right),$$

$$G_-^{(1)} = \log(8\pi) - 3 + \frac{7\alpha\zeta(3)}{\pi^2} - \frac{\alpha}{2}\left(\mathrm{u}_1(\alpha) + \alpha \mathrm{u}_3(\alpha)\right).$$

$$G_+^{(2)} = 7\zeta(3) + 8\alpha\left[\frac{1}{3} - \frac{7\zeta(3)}{\pi^2}\right] + \frac{4\alpha^2}{\pi^2}\left[-\frac{28\zeta(3)}{3} + \frac{49\zeta^2(3)}{\pi^2} + \frac{62\zeta(5)}{\pi^2}\right] +$$

$$+\left[-\frac{\pi^2}{2} + 4\alpha + 4\alpha^2 - \frac{28\alpha^2\zeta(3)}{\pi^2}\right]\alpha \mathrm{u}_3(\alpha) + \alpha^4 \mathrm{u}_3^2(\alpha) - 4\alpha^3 \mathrm{u}_5(\alpha),$$

$$G_-^{(2)} = 2 - \frac{4\alpha}{3} + \frac{4\alpha^2}{\pi^2}\left[\frac{14}{3}\zeta(3) - \frac{31}{\pi^2}\zeta(5)\right] - 2\alpha^3 \mathrm{u}_3(\alpha) + 2\alpha^3 \mathrm{u}_5(\alpha),$$

where

$$\mathrm{u}_{2k-1}(\alpha) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \frac{\sinh^2 t}{t\cosh^{2k-1} t\left(\alpha\sinh t + t\cosh t\right)}\,dt.$$

Extracting the spectral data

For $\beta \neq 0$ the Wronskian $W(0)$ does not factorize

$$W(0) = Q_+(i)Q_-(-i) - Q_+(-i)Q_-(i) = 2i\frac{\gamma}{1+\alpha}\frac{\alpha^2}{\alpha^2-\beta^2},$$
$$\gamma = \sqrt{(1+\alpha+\beta)(1+\alpha-\beta)}$$

since $Q_\pm(\nu|\beta)$ satisfy more general symmetry properties

$$Q_\pm(\nu|\beta) = \mp Q_\pm(-\nu|-\beta).$$

But there are two surprisingly simple linear relations between these four functions. Namely, if we define the linear combinations

$$\mathbf{Q}_-(\nu) = Q_-(\nu) - \frac{i\beta\pi^2\lambda}{2(1+\alpha)(1+\alpha+\gamma)}Q_+(\nu);$$
$$\mathbf{Q}_+(\nu) = Q_+(\nu) + \frac{2i\beta}{\pi^2\lambda}\frac{1+\alpha}{1+\alpha+\gamma}Q_-(\nu)$$

then

$$\frac{\mathbf{Q}_-(i)}{\mathbf{Q}_-(-i)} = \frac{\alpha-\beta}{\alpha+\beta}, \quad \frac{\mathbf{Q}_+(i)}{\mathbf{Q}_+(-i)} = \frac{\beta+\pi^2\lambda}{\beta-\pi^2\lambda}\frac{\alpha-\beta}{\alpha+\beta}.$$

Extracting the spectral data

In terms of $\mathbf{Q}_\pm(i)$ the Wronskian factorizes

$$W(0) = \frac{\pi^2\lambda}{\beta+\pi^2\lambda}\frac{\alpha+\beta}{\alpha-\beta}\frac{1+\alpha+\gamma}{\gamma}\mathbf{Q}_+(i)\mathbf{Q}_-(i).$$

Then, we conjecture the following formulas:

$$\mathbf{Q}_-(i) = \frac{\alpha}{\alpha+\beta}\frac{\gamma}{1+\alpha}\frac{\mathcal{D}_+(\lambda)}{\mathcal{D}_-(\lambda)},$$
$$\mathbf{Q}_+(i) = 2i\frac{\beta+\pi^2\lambda}{\pi^2\lambda}\frac{\alpha}{\alpha+\beta}\frac{\gamma}{1+\alpha+\gamma}\frac{\mathcal{D}_-(\lambda)}{\mathcal{D}_+(\lambda)}.$$

and log-derivative relations

$$\partial_\lambda \log \mathcal{D}_-(\lambda) + (\dots)\frac{1}{\lambda} + (\dots) = 2i\left(1+\frac{\beta}{\pi^2\lambda}\right)\partial_\nu \log \mathbf{Q}_+(\nu)\Big|_{\nu=i},$$

$$\partial_\lambda \log \mathcal{D}_+(\lambda) + (\dots)\frac{1}{\lambda} + (\dots) = 2i\left(1-\frac{\alpha}{\pi^2\lambda}\right)\partial_\nu \log \mathbf{Q}_-(\nu)\Big|_{\nu=i} - \\ -2i\frac{\alpha+\beta}{\pi^2\lambda}\partial_\nu \log \mathbf{Q}_-(-\nu)\Big|_{\nu=i}.$$

Extracting the spectral data

Using these relations, we obtain

$$G_{-}^{(1)} = \log 8\pi - 2 + \frac{\pi}{4\beta^2 g_1}[\pi(\alpha+\beta) - \pi\alpha g_1 + 2\beta\xi_1] - \frac{(\alpha+\beta)^2 \mathtt{i}_2(\alpha+\beta) - (\alpha-\beta)^2 \mathtt{i}_2(\alpha-\beta)}{8\beta},$$

$$G_{+}^{(1)} = \log 8\pi - 2 - \frac{\pi}{4\beta^2 g_1}[\pi(\alpha+\beta) - \pi\alpha g_1 + 2\beta\xi_1] - \frac{(\alpha+\beta)^2 \mathtt{i}_2(\alpha+\beta) - (\alpha-\beta)^2 \mathtt{i}_2(\alpha-\beta)}{8\beta},$$

$$G_{-}^{(2)} = \frac{\pi^2}{16\beta^4 g_1^2 g_2}\Bigg[(\pi(\alpha+\beta)+2\beta\xi_1)^2 g_2 - 8\pi\alpha(\pi(\alpha+\beta)+2\beta\xi_1)g_1 g_2 + 16(\pi\alpha+\beta\xi_2)(\pi(\alpha+\beta)+\beta\xi_2)g_1^2 + \left(4(\pi^2+2\alpha)\beta^2 - 9\pi^2\alpha^2\right)g_1^2 g_2\Bigg] - \frac{\pi^2(\alpha^2-\beta^2)}{16\beta^3}\Big((\alpha+\beta)\mathtt{i}_2(\alpha+\beta) - (\alpha-\beta)\mathtt{i}_2(\alpha-\beta)\Big),$$

$$G_{+}^{(2)} = \frac{\pi^2}{16\beta^4 g_1^2 g_2}\Bigg[(\pi(\alpha+\beta)+2\beta\xi_1)^2 g_2 + 8\pi\alpha(\pi(\alpha+\beta)+2\beta\xi_1)g_1 g_2 - 16(\pi\alpha+\beta\xi_2)(\pi(\alpha+\beta)+\beta\xi_2)g_1^2 - \left(4(\pi^2-2\alpha)\beta^2 - 7\pi^2\alpha^2\right)g_1^2 g_2\Bigg] - \frac{\pi^2(\alpha^2-\beta^2)}{16\beta^3}\Big((\alpha+\beta)\mathtt{i}_2(\alpha+\beta) - (\alpha-\beta)\mathtt{i}_2(\alpha-\beta)\Big),$$

etc, where

$$\xi_k = \arctan\frac{\pi k}{2\beta}, \quad g_k \stackrel{\text{def}}{=} G(\nu_k), \quad G(\nu|\alpha,\beta) \stackrel{\text{def}}{=} \frac{S_0(\nu|\alpha+\beta)}{S_0(\nu|\alpha-\beta)},$$

$$S_0(\nu+i|\alpha) = e^{\frac{i}{4}\int_{-\infty}^{\infty}\log\left(\frac{4\pi\tanh\left(\frac{\pi t}{2}\right)}{t+\frac{2\alpha}{\pi}\tanh\left(\frac{\pi t}{2}\right)}\right)\left(\tanh\frac{\pi(t-\nu)}{2} - \tanh\frac{\pi t}{2}\right)dt}, \quad \mathtt{i}_{2k}(\alpha) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty}\frac{\sinh t(\sinh 2t - 2t)}{t\cosh^{2k} t\left(\alpha\sinh t + t\cosh t\right)}dt.$$

Extracting the spectral data

For the spectral problem

$$\psi(u) - 2i\beta \fint_{\mathbb{R}} S(u,v)\psi(v)dv = \lambda \int_{\mathbb{R}} K(u,v)\psi(v)dv,$$

where

$$K(u,v) = \frac{\pi(u-v)}{2\sinh\frac{\pi(u-v)}{2}}\frac{1}{f(v)}, \quad S(u,v) = \frac{\pi}{2\sinh\frac{\pi(u-v)}{2}}\frac{1}{f(v)}, \quad f(v) = \frac{2\alpha}{\pi} + v\coth\frac{\pi v}{2}$$

we were able to

1. Find the solution of the corresponding inhomogeneous problem (perturbative in λ, but exact in α and β)

$$\psi_\pm(u) - 2i\beta \fint_{\mathbb{R}} S(u,v)\psi_\pm(v)dv = \lambda \int_{\mathbb{R}} K(u,v)\psi_\pm(v)dv + e_\pm(u),$$

$$e_+(v) = \frac{v}{\sinh\frac{\pi v}{2}}, \quad e_-(v) = \frac{1}{\sinh\frac{\pi v}{2}}$$

2. Relate the spectral determinants $\mathcal{D}_\pm(\lambda)$ to $Q_\pm(i)$, $Q'_\pm(i)$ etc and thus find the spectral sums explicitly

The same holds for Ising like BS-equation with

$$f(v) = \frac{2\alpha}{\pi} + v\tanh\frac{\pi v}{2}, \quad e_+(v) = \frac{1}{\cosh\frac{\pi v}{2}}, \quad e_-(v) = \frac{v}{\cosh\frac{\pi v}{2}}$$

Conclusion

For $\beta = 0$ there are more relations

$$
\begin{aligned}
&Q_+(i) = i\frac{\mathcal{D}_-(\lambda)}{\mathcal{D}_+(\lambda)},\\
&\frac{Q'_+(i)}{Q_+(i)} + (\dots) = -\frac{i}{2}\partial_\lambda \log \mathcal{D}_-(\lambda),\\
&i\left(1 - \frac{2\alpha}{\pi}\lambda^{-1}\right)\frac{Q''_+(i)}{Q_+(i)} - \frac{1}{2\lambda}\frac{Q'_+(i)}{Q_+(i)} - i\left(\frac{Q'_+(i)}{Q_+(i)}\right)^2 + (\dots) = -\frac{i}{4}\partial^2_\lambda \log \mathcal{D}_-(\lambda),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots
\end{aligned}
$$

etc

Do they generalize for $\beta \neq 0$?

What is the most general class of integrable kernels with such properties?

Can one systematically study $1/N_c$ corrections?